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Complex temporal association in neural networks

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Abstract. Models of temporal association in neural networks are generalised to provide mechanisms for decision making and loop control. Systems equipped with these capabilities can handle situations where a succession of states is not unambiguously defined and the system has to choose which out of several, in principle equivalent, paths it is going to follow. The choice is made on the basis of past experience of the network. No short-term synaptic plasticity is needed.

1. Introduction

The collective behaviour of neural networks with symmetric synaptic interconnections is by now rather well understood [1–8]. It has been interpreted as a relaxational motion in the free energy landscape associated with an energy function or Hamiltonian:

$$H_N = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j \quad (1)$$

with symmetric couplings $J_{ij} = J_{ji}$. As time proceeds, the dynamics converges to a *static* equilibrium. Data can be stored in the couplings in that by a suitable choice of the J_{ij} several specific firing patterns $\{\xi_i^\nu; 1 \leq i \leq N\}$, $1 \leq \nu \leq q$, of the formal neurons $S_i = \pm 1$ can be made stable states of the Hamiltonian (1).

More recently, issues of *temporal* association in neural nets, such as those involved in counting, in reciting poems, and in the control of motion, have been raised [9–17]. Since temporal association is evidently incompatible with the conditions of static equilibrium, it had been realised rather early [2, 9] that, to achieve transitions between a given set of nominated patterns of the network, a certain amount of asymmetry of the synaptic connections is needed. On the basis of this idea, several models of sequence generators have been proposed. These can roughly be divided into two categories: models that rely on short-term synaptic plasticity [9, 10] and models that do not [11–17].

In most of the sequence generators so far considered, asymmetric interactions are used to encode transitions between *pairs* of embedded patterns. Temporal association in these models is restricted to situations where a succession of states is always unambiguously defined, i.e. to *linear* or *cyclic* sequences or to sequences *down* a tree structure [11–16]. In the present paper we devise mechanisms by which such a restriction can be removed. In so doing, we opt to avoid introducing short-term synaptic plasticity, since models with fixed synapses are usually much simpler and therefore easier to analyse. Moreover, they are also more easily adapted to situations which are

different from those for which they were designed in the first place. For a complementary approach, which *does* rely on synaptic plasticity during retrieval, we refer to Dehaene *et al* [10].

Our method of storing and recalling complex sequences generalises the models for 'linear' temporal association of Kleinfeld [11] and Sompolinsky and Kanter [12] by introducing synapses with *multiple* time delays. Depending on the delay *mechanism*, there are two basically different approaches, which we shall discuss in turn, one relying on three- (or higher-order) neuron interactions and another doing without them.

A related approach to the generation of simple and certain complex sequences in neural networks operating with parallel dynamics was recently proposed by Personnaz *et al* [17]. They use pseudo-inverse solutions, as we shall do below when dealing with correlated patterns, and a concatenation-of-states technique instead of time delays to deal with certain ambiguities. Let us further mention that complex sequences in the context of matrix memories and neural networks had, in fact, already been considered by Kohonen [18] and Labos [19] several years before the current phase of research activities on sequence generation in neural nets began with work of Peretto and Niez [9].

The number of possible realisations of complex sequences is clearly without bound and there is no way of explicitly dealing with all of them. We shall therefore illustrate the salient features of our approach by way of reference to an example, namely the problem of encoding and recalling a succession of states consisting of several disjoint linear pattern sequences, separated from each other by yet another common and recurring sequence of states. Such a problem would arise, for instance, in singing a song of many verses, separated from each other by a chorus. We shall present two basically different realisations, discussing the dynamics of the version relying on three-neuron interaction in some detail and only hint at the modifications and simplifications for the second. We then extract from our example what we take to constitute the *syntax* of our approach to complex association and discuss the role and range of the various parameters of our model. Finally, we generalise our theory to deal with complex sequences of *correlated* patterns and present a discussion.

2. The model: three-neuron interactions

We consider a network of N neurons in which the *elementary* linear sequences P_ρ , $0 \leq \rho \leq r$, are to be stored, each sequence P_ρ consisting of a succession of states (ρ, μ) , with $1 \leq \mu \leq p_\rho$. We require the sequences to be recalled in the order $P_1 P_0 P_2 P_0 \dots P_{r+1}$ so that there is an ambiguity whenever the system is in the state $(0, p_0)$. In a song, P_1, P_2, \dots , would be the verses and P_0 the chorus. By identifying P_{r+1} and P_1 , we get a complex cycle with repetitions. We denote by $\{\xi_i^{\rho, \mu}; 1 \leq i \leq N\}$ the pattern (ρ, μ) , which for the moment we assume to be random and unbiased, so that the $\xi_i^{\rho, \mu}$ take the values ± 1 with equal probability.

Our method to store the above-mentioned complex sequence is based on models for linear sequence generation proposed by Kleinfeld [11] and Sompolinsky and Kanter [12], supplemented by mechanisms to resolve the ambiguity that occurs at the end of P_0 . To encode the complex sequence, we need *three* kinds of synapse. The synapses of the first kind are symmetric [2]:

$$J_{ij}^{(1)} = N^{-1} \sum_{(\rho, \mu)} \xi_i^{\rho, \mu} \xi_j^{\rho, \mu} \quad (2)$$

and are used to stabilise the individual firing patterns of the complex sequence. The synapses of the second kind [2, 11, 12]:

$$J_{ij}^{(2)} = \varepsilon N^{-1} \sum'_{(\rho, \mu)} \xi_i^{\rho, \mu \oplus 1} \xi_j^{\rho, \mu} \quad (3)$$

are asymmetric and generate transitions between successive states, provided that 'succession' is unambiguously defined. The primed sum in (3) therefore excludes the state $(0, p_0)$, the 'last note of the chorus'. Furthermore, we define

$$(\rho, \mu \oplus 1) = \begin{cases} (\rho, \mu + 1) & \text{if } \mu < p_\rho \\ (0, 1) & \text{if } \mu = p_\rho \text{ and } \rho \neq 0 \end{cases} \quad (4)$$

so that P_0 is entered after each P_ρ , $\rho \neq 0$. To encode the proper succession of the linear subsequences, a third kind of asymmetric synapse is needed. In the first approach to be discussed here, this synapse embodies a *three*-neuron interaction, namely

$$J_{ijk}^{(3)} = \tilde{\varepsilon} N^{-2} \sum_{\rho=1}^r \xi_i^{\rho+1, 1} \xi_j^{0, p_0} \xi_k^{\rho, p_\rho}. \quad (5)$$

We turn to the other approach in § 4.

As in the models for linear temporal association [11–15], the synapses $J_{ij}^{(2)}$ and $J_{ijk}^{(3)}$ must be endowed with their own characteristic response times in order to recall the encoded complex sequence in a controlled fashion. In particular, two different response times are needed for the two input channels j and k of $J_{ijk}^{(3)}$: a short one (τ_a), associated with j , to initiate a transition from $(0, p_0)$ to one of the P_ρ , $\rho \neq 0$, and a longer one (τ_b) associated with k to 'remind' the system of the previous P_ρ and thus provide the necessary information as to which P_ρ comes next. We will thus take the local field (postsynaptic potential) experienced by neuron i to be given by

$$\begin{aligned} h_i(t) &= h_i^{(1)}(t) + h_i^{(2)}(t) + h_i^{(3)}(t) \\ &= \sum_{j(i \neq i)} J_{ij}^{(1)} S_j(t) + \sum_j J_{ij}^{(2)} \bar{S}_j^a(t) + \sum_{j,k} J_{ijk}^{(3)} \bar{S}_j^a(t) \bar{S}_k^b(t). \end{aligned} \quad (6)$$

Here $\bar{S}_j^\Delta(t)$, $\Delta = a, b$, denotes a convolution of $S_j(t)$ with a non-negative memory kernel $w_\Delta(t)$ characterised by an internal time constant τ_Δ [11–13]:

$$\bar{S}_j^\Delta(t) = \int_{-\infty}^t ds w_\Delta(t-s) S_j(s) \quad \int_0^\infty ds w_\Delta(s) = 1 \quad \Delta = a, b. \quad (7)$$

Various choices for the memory kernels $w_\Delta(t)$ may be considered and were previously discussed in the literature, the most popular being δ function, step function and exponential delay [11–15]. If the dynamics of the system is defined by the threshold condition

$$S_i(t + \Delta t) = \text{sgn}[h_i(t)] \quad (8)$$

or a probabilistic version thereof, either sequentially or in parallel, all three memory functions give rise to stable complex sequences. For parallel dynamics, one may put $\Delta t = 1$ in (8). For sequential dynamics, Δt should scale with system size N as $\Delta t \propto N^{-1}$. For the sake of definiteness, the ensuing discussion of the dynamics will be phrased in terms of exponential delay. In passing we note that the inclusion of 'delayed self-excitations' through $h_i^{(2)}(t)$ and $h_i^{(3)}(t)$ is intentional (though immaterial in the low

loading limit). The purpose of these contributions to the local field is to induce transitions and decision making, on the basis of past experience of the whole system.

Before proceeding to a discussion of the dynamics, let us note that, for instance in the case of δ -function delays, one can dispense with the symmetric synapses (2) [15, 20]. For general delay mechanisms, however, they *are* necessary to stabilise the individual patterns of the sequence and thereby to prevent that the sequence gets washed out as time proceeds.

3. Dynamics

The dynamics of the network is most easily explained in the limit where $q/N = N^{-1} \sum_{\rho=0}^r p_{\rho} \rightarrow 0$, as $N \rightarrow \infty$. In this limit, the local field in (6) may be expressed in terms of the overlaps $m_{\rho,\mu}(t) = N^{-1} \sum_i \xi_i^{\rho,\mu} S_i(t)$ as

$$h_i(t) = \sum_{(\rho,\mu)} \xi_i^{\rho,\mu} m_{\rho,\mu}(t) + \varepsilon \sum_{(\rho,\mu)} \xi_i^{\rho,\mu \oplus 1} \bar{m}_{\rho,\mu}^a(t) + \tilde{\varepsilon} \sum_{\rho=1}^r \xi_i^{\rho+1,1} \bar{m}_{0,p_0}^a(t) \bar{m}_{\rho,p_{\rho}}^b(t). \tag{9}$$

The $\bar{m}_{\rho,\mu}^{\Delta}(t)$ denote a convolution of $m_{\rho,\mu}(t)$ with the memory kernel $w_{\Delta}(t)$ as in (7). Each of the $\bar{m}_{\rho,\mu}^{\Delta}(t)$ builds up while the system is in the corresponding state (ρ, μ) , becomes of order one, if the system has stayed there for a time of order τ_{Δ} , and decays on a timescale of the same order of magnitude, once the system has left (ρ, μ) . This mechanism induces transitions between successive states belonging to the unambiguously connected linear subparts $P_{\rho}P_0$ of the complex sequence at regular time intervals of order τ_a on the basis of the contributions of $h_i^{(1)}(t)$ and $h_i^{(2)}(t)$ to the local field alone, as is discussed at length in [11-15].

Assuming now that the system has entered the linear subsequence P_0 (the chorus) via the state (γ, p_{γ}) , for some $1 \leq \gamma \leq r$, we see that $h_i^{(3)}(t)$ becomes sizeable only when the system has traversed P_0 and spent a time of order τ_a in the state $(0, p_0)$, *provided* $\tau_b > \tau_a$ is chosen such that some memory $\bar{m}_{\gamma,p_{\gamma}}^b > 0$ of the last pattern of the sequence P_{γ} which immediately preceded P_0 is still present. If, moreover, $\bar{m}_{\gamma,p_{\gamma}}^b$ dominates the $\bar{m}_{\rho,p_{\rho}}^b$ for $\rho \neq \gamma$ (again a matter of tuning τ_b) and $\tilde{\varepsilon}$ is large enough, then $h_i^{(3)}(t)$ will induce a transition to the state $(\gamma + 1, 1)$, the first pattern of the subsequence $P_{\gamma+1}$, which was meant to follow $P_{\gamma}P_0$. Immediately after the transition, $h_i^{(3)}(t)$ still favours $(\gamma + 1, 1)$ for a time of order τ_a during which this contribution to the local field decays, but becomes negligible once the transition into $(\gamma + 1, 2)$ is achieved. Only after the system has completed the next connected linear subpart $P_{\gamma+1}P_0$ does $h_i^{(3)}(t)$ become sizeable again and will, according to the foregoing discussion, induce a transition into $(\gamma + 2, 1)$; and so on.

We have verified the above scenario by solving (8) directly in the thermodynamic limit, using the exact dynamic evolution equations for the overlaps $m_{\rho,\mu}(t)$ as derived by Riedel *et al* [15]. The equations are

$$m_{\rho,\mu}(t+1) = \sum_{\mathbf{x}} p(\mathbf{x}) x^{\rho,\mu} \tanh[\beta h(\mathbf{x}, t)] \tag{10}$$

for parallel dynamics, and

$$\dot{m}_{\rho,\mu}(t) = - \left(m_{\rho,\mu} - \sum_{\mathbf{x}} p(\mathbf{x}) x^{\rho,\mu} \tanh[\beta h(\mathbf{x}, t)] \right) \tag{11}$$

for sequential dynamics. Here \mathbf{x} ranges through the 2^q corners of the hypercube $C^q = [-1, 1]^q$, the range of the random vectors $\xi_i = (\xi_i^{\rho,\mu}; 0 \leq \rho \leq r, 1 \leq \mu \leq p_\rho)$, $p(\mathbf{x}) = \text{Prob}\{\xi_i = \mathbf{x}\}$ [7, 21, 22], and

$$h(\mathbf{x}, t) = \sum_{(\rho,\mu)} x^{\rho,\mu} m_{\rho,\mu}(t) + \varepsilon \sum'_{(\rho,\mu)} x^{\rho,\mu \oplus 1} \bar{m}_{\rho,\mu}^a(t) + \tilde{\varepsilon} \sum_{\rho=1}^r x^{\rho+1,1} \bar{m}_{0,\rho_0}^a(t) \bar{m}_{\rho,\rho_\rho}^b(t) \quad (12)$$

is the local field on the sublattice $I(\mathbf{x})$ which consists of those sites i where $\xi_i = \mathbf{x}$. As usual, β denotes the inverse temperature and \dot{m} is a differentiation of m with respect to time. For details, the reader may consult [15, 23].

Figure 1 presents a solution of (10) for a complex cycle of the form $P_1 P_0 P_2 P_0 P_1 \dots$, where each of the elementary sequences consists of three states. In figure 2 an example is shown where a three-neuron interaction is used to provide loop control: the system loops around $P_1 = \{(1, \mu); \mu = 1, \dots, 4\}$ twice, then escapes into $P_2 = \{(2, \mu); \mu = 1, \dots, 5\}$, enters P_1 via (1, 4) again, and so on. The three-neuron interaction is chosen so that its contribution to the local field is $h_i^{(3)}(t) = \tilde{\varepsilon} (\xi_i^{2,1} - \xi_i^{1,4}) \bar{m}_{1,4}^a(t) \bar{m}_{1,3}^b(t)$, and $J_{ij}^{(2)}$ has no term that would induce a transition (1, 4) \rightarrow (2, 1). The rationale behind this choice of $h_i^{(3)}(t)$ is that it switches off the 'loop-closing' transition term (1, 4) \rightarrow (1, 1) encoded in $J_{ij}^{(2)}$ and simultaneously switches on a transition term (1, 4) \rightarrow (2, 1) that causes the system to leave the loop at a specified time controlled by $w_b(t)$.

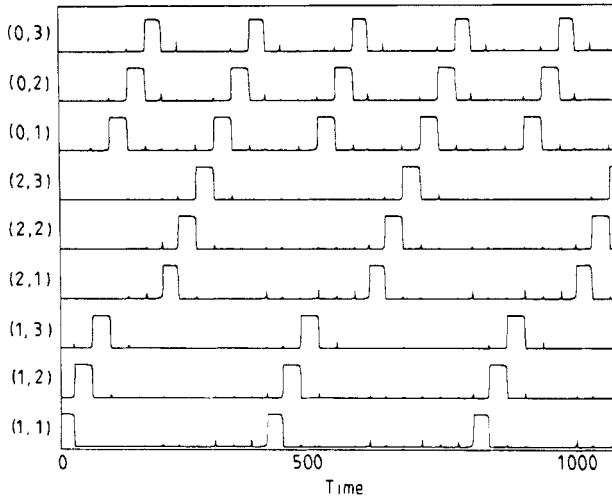


Figure 1. Solution to (10) for the complex cycle $P_1 P_0 P_2 P_0 \dots$ where P_1 and P_2 are 'verses' and P_0 is a 'chorus'. Each trace represents the time evolution of an overlap $m_{\rho,\mu}(t)$ with one of the embedded patterns (ρ, μ) . The memory kernels w_a and w_b decay exponentially with $\tau_a = 15$ and $\tau_b = p_0 \tau_a = 45$ iterations. The inverse temperature is $\beta = 10$, the asymmetry parameters are $\varepsilon = 1$ and $\tilde{\varepsilon} = 17.75$, respectively.

Numerical simulations show that the above analysis still holds, i.e. stable complex cycles still exist—when $\alpha = q/N$ is finite and the random overlaps between the patterns cannot be ignored. The only proviso is that α is not too big. Our numerical work also confirms the stability of small complex cycles in the presence of extensively many other patterns outside the cycle, in agreement with an analytical treatment from [15].

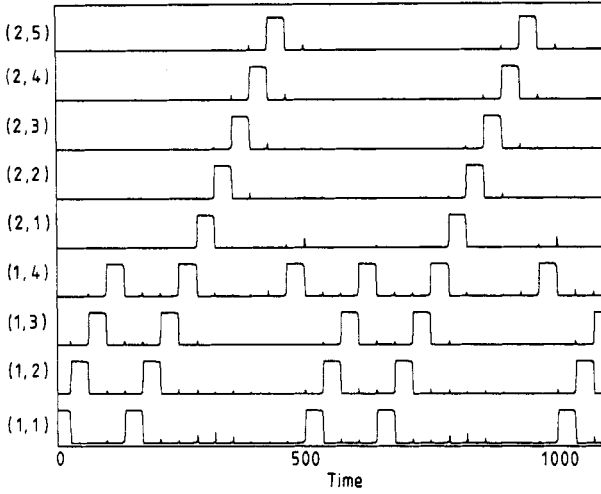


Figure 2. Solution to (10) for a system with three-neuron interactions providing *loop control*. The memory kernel w_a and the parameters τ_a, β and ε are as in figure 1. Here, however $\tilde{\varepsilon} = 1$ and w_b is a δ -function delay with $\tau_b = 215$ iterations.

4. Two-neuron interactions

So far our analysis has been entirely within an approach that relies on three-neuron interactions. One might object that in a fully connected system this approach would raise severe wiring problems. Serious though the objection is, one may circumvent it by introducing dilution in the manner of [8]. Our analytical theory [15] is able to deal with it. We have as yet, however, not undertaken any quantitative studies of this point in the present context. A solution in a completely different direction is possible, if we use delay mechanisms, such as δ -function delay, which have memory functions with *support bounded away from zero*. Using this type of delay, one may replace $J_{ijk}^{(3)}$ in (5) by

$$\tilde{J}_{ij}^{(2)} = \tilde{\varepsilon} N^{-1} \sum_{\rho=1}^r \xi_i^{\rho+1,1} \xi_j^{\rho,p_\rho} \tag{13}$$

and take the local field experienced by neuron i to be

$$h_i(t) = \sum_{j(\neq i)} J_{ij}^{(1)} S_j(t) + \sum_j J_{ij}^{(2)} \bar{S}_j^a(t) + \sum_j \tilde{J}_{ij}^{(2)} \bar{S}_j^b(t) \\ = \sum_{(\rho,\mu)} \xi_i^{\rho,\mu} m_{\rho,\mu}(t) + \varepsilon \sum_{(\rho,\mu)} \xi_i^{\rho,\mu \oplus 1} \bar{m}_{\rho,\mu}^a(t) + \tilde{\varepsilon} \sum_{\rho=1}^r \xi_i^{\rho+1,1} \bar{m}_{\rho,p_\rho}^b(t). \tag{14}$$

Equation (14) now replaces (6) and (9). We have $w_b(t) = \delta(t - \tau_b)$ with $\tau_b = (p_0 + 1)t_0$, and t_0 denoting the time the system spends in each state (see § 6). In the present setup, the mechanism that induces the transition from $(0, p_0)$ to some $(\gamma + 1, 1)$ is quite different from what it was before. It is *exclusively* controlled by the state (γ, p_γ) which immediately preceded the ‘chorus’ P_0 and the mechanism is the same as in the standard sequence generators [11–15].

Unlike before, the transition is not triggered by $(0, p_0)$ in the sense that the system having been in $(0, p_0)$ for a specified time t_0 is *immaterial* for the transition into $(\gamma + 1, 1)$ to occur. What is relevant, so to speak, is the last note of the preceding verse rather

than that of the chorus. This ‘non-locality in time’ aspect may seem somewhat implausible. It can be avoided by adding weak direct transition terms from $(0, p_0)$ to all of the $(\rho, 1)$, $\rho \neq 0$,

$$J_{ij}^0 = \varepsilon_0 N^{-1} \left(\sum_{\rho=1}^r \xi_i^{\rho,1} \right) \xi_j^{0,p_0} \quad (15)$$

and letting the local field at i be given by

$$h_i(t) = \sum_{(\rho,\mu)} \xi_i^{\rho,\mu} m_{\rho,\mu}(t) + \varepsilon \sum'_{(\rho,\mu)} \xi_i^{\rho,\mu \oplus 1} \bar{m}_{\rho,\mu}^a(t) + \varepsilon_0 \left(\sum_{\rho=1}^r \xi_i^{\rho,1} \right) \bar{m}_{0,p_0}^a(t) + \bar{\varepsilon} \sum_{\rho=1}^r \xi_i^{\rho+1,1} \bar{m}_{\rho,p_\rho}^b(t) \quad (16)$$

with $w_b(\tau) = \delta(t - \tau_b)$ as above. Given the inverse temperature β , ε_0 and $\bar{\varepsilon}$ may be chosen small enough so that neither the third nor the fourth term on the right-hand side of (16) would induce any transition by itself, and large enough so that the combination does. The choice of the final state depends on the ‘verse’ P_γ that preceded P_0 in that the fourth term selects $(\gamma + 1, 1)$. Using standard arguments [11–15], the reader can easily convince himself that, for instance, in the case of parallel dynamics at $T = 0$ the bounds $\varepsilon_0 < r^{-1}$, $\bar{\varepsilon} < 1$ and $r\varepsilon_0 + \bar{\varepsilon} > 1$ will ensure that the complex sequence generator (16) functions as desired. We have checked this both by solving (10) and by numerical simulations.

5. Syntax

Stepping back for an overview, one may ask what is the essence or *syntax* of our approach to complex association that is needed to tackle problems different from the examples presented above. The answer simply is that it is contained in the examples, and may therefore be extracted from them. This is what we now proceed to do.

In all cases the problem is to deal with an ambiguity of the following form. Given a network state 0, there are several possible paths P_ρ , $1 \leq \rho \leq r$, which start here. Let us denote the entrance states of these paths by ρ , where ρ ranges through \mathbf{R} . There are thus several possible transitions $0 \rightarrow \rho$ with 0 as the initial state. Once the system is in 0, extra information is necessary for the decision as to which P_ρ is to be taken (which transition $0 \rightarrow \rho$ is to be initiated). This piece of extra information is provided by past experience of the network. A subset \mathbf{G} of the set of embedded states is selected to function as ‘control states’ as follows. The fact that one of them was visited at a specified time before the decision where to go to starting from 0 is on the agenda determines which of the possible target states ρ , $\rho \in \mathbf{R}$, is to be the final state of the transition.

In the two-neuron interaction approach, this is achieved by introducing synapses of the form

$$\tilde{J}_{ij}^{(2)} = \tilde{\varepsilon} N^{-1} \sum_{\rho,\gamma} \xi_i^\rho W_{\rho,\gamma} \xi_j^\gamma \quad (17)$$

where ρ and γ range through the sets \mathbf{R} and \mathbf{G} of target and control states, respectively. In the song example of the previous sections, these sets are given by the first and last notes of the verses. In (17), W is a matrix whose elements $W_{\rho,\gamma}$ are zero or one, specifying which of the target states is selected by each of the control states. Each

term in (17) may, if necessary, be endowed with its own characteristic response time $\tau_{\rho\gamma}$ so that the contribution of (17) to the local field is

$$\tilde{h}_i^{(2)} = \tilde{\varepsilon} \sum_{\rho,\gamma} \xi_i^\rho W_{\rho,\gamma} m_\gamma(t - \tau_{\rho\gamma}) \quad (18)$$

where $\tau_{\rho\gamma}$ may, but need not, vary with ρ and γ . As discussed in the previous section, one may in addition introduce weak direct transition terms from 0 to all of the target states ρ so that transitions are actually only induced by combining them with (17) and (18).

In a three-neuron interaction approach, one would have synapses of the form

$$J_{ijk}^{(3)} = \tilde{\varepsilon} N^{-2} \sum_{\rho,\gamma} \xi_i^\rho W_{\rho,\gamma} \xi_j^\rho \xi_k^\gamma \quad (19)$$

and

$$h_i^{(3)} = \tilde{\varepsilon} \sum_{\rho,\gamma} \xi_i^\rho W_{\rho,\gamma} \bar{m}_\rho^\alpha(t) \bar{m}_\gamma^{\rho\gamma}(t) \quad (20)$$

instead of (17) and (18).

If the number $|\mathbf{R}|$ of target states equals the number $|\mathbf{G}|$ of control states, then the matrix W is a permutation matrix. If $|\mathbf{R}| < |\mathbf{G}|$, some of the control states may select one and the same target state. On the other hand, for $|\mathbf{R}| > |\mathbf{G}|$ some of the control states are responsible for more than one transition. This can (and must) be achieved by endowing different transition terms associated with such control states with different response time $\tau_{\rho\gamma}$. A trivial application of this technique (without the aspect of path selection) is already contained in the song example of the previous sections, where the states (γ, p_γ) controlled the transition into $(0, 1)$, using the delay τ_a , and into $(\gamma + 1, 1)$, using the delay τ_b .

Note that the problem of loop control is covered by the above general considerations. Here, each time the final state of the loop is encountered, the question is whether to leave the loop or to stay there.

6. How to choose the parameters

The performance of the complex sequence generators discussed in the previous sections depends on various parameters, namely the inverse temperature β , which quantifies the level of stochastic noise in the system, the decay times τ_a and τ_b of the memory functions w_a and w_b and the asymmetry parameters ε and $\tilde{\varepsilon}$ and in (15) and (16) also ε_0 . Clearly, the problem how to choose these parameters cannot be discussed without reference to the *topology* of the complex sequence itself. To illustrate the general case, we shall therefore return to the song paradigm of the previous sections.

The problem of choosing parameters can be divided into two simple subproblems. As far as the dynamics on the unambiguously connected linear subsequences $P_\rho P_0$ is concerned, it is, as usual [11–15], completely determined by the parameters β , τ_a , and ε alone. In particular, the persistence time t_0 [12], the time the system spends in each state $(\rho, \mu) \neq (0, p_0)$, is a function of β , τ_a and ε . Once t_0 is known, the dynamics of the transitions $(0, p_0) \rightarrow (\rho, 1)$ at the branch point $(0, p_0)$ can be discussed. Given t_0 , the time t^* the system spends in the branch point state $(0, p_0)$ is determined by the remaining parameters τ_b and $\tilde{\varepsilon}$ (and, possibly, ε_0). It can be evaluated through arguments analogous to the ones used to compute t_0 .

The persistence time t_0 for the linear case was calculated by Sompolinsky and Kanter in the zero-temperature limit [12]. Since our evaluation of t^* draws upon the same set of ideas, we shall briefly repeat the argument of [12] and give a heuristic generalisation of it to non-zero (but low) temperatures.

Let us consider a standard sequence generator [11–15] operating with a set of symmetric synapses $J_{ij}^{(1)}$ and a set of asymmetric synapses $J_{ij}^{(2)}$, which are endowed with a slow dynamic response characterised by a memory function $w_a(t)$; cf (2) and (3). We assume that transitions between successive states of the sequence (here, for simplicity, denoted by μ) have occurred regularly in time with period t_0 . We then consider the local field at i when the system has been in state μ for a time δ :

$$h_i(t) = \xi_i^\mu + \varepsilon [\xi_i^{\mu+1} W_a(0, \delta) + \xi_i^\mu W_a(\delta, \delta + t_0) + \xi_i^{\mu-1} W_a(\delta + t_0, \delta + 2t_0) \dots] \quad (21)$$

where

$$W_a(t_1, t_2) = \int_{t_1}^{t_2} w_a(t) dt. \quad (22)$$

According to [12], the transition from μ to $\mu + 1$ starts on the special sublattice

$$I_\mu = \{i; \xi_i^\nu = -\xi_i^\mu \text{ for all } \nu \neq \mu\} \quad (23)$$

where the spins have been parallel to $\xi_i^{\mu+1}$ all the time *except* when the system was in state μ . On this sublattice, the local field is biased most strongly in favour of $\xi_i^{\mu+1}$ and the transition is imagined to start when $h_i \xi_i^\mu$ becomes negative on I_μ . The corresponding δ must be identified with t_0 . If the sequence is of sufficient length, so that we have $W_a(0, \delta) + W_a(\delta, \delta + t_0) + W_a(\delta + t_0, \delta + 2t_0) + \dots \approx 1$ in (20), and if transients are neglected, this gives the following equation connection t_0 , ε and τ_a :

$$\begin{aligned} 0 &= h_i \xi_i^\mu = 1 + \varepsilon W_a(t_0, 2t_0) - \varepsilon [W_a(0, t_0) + W_a(2t_0, 3t_0) + \dots] \\ &\approx 1 - \varepsilon + 2\varepsilon W_a(t_0, 2t_0) \end{aligned} \quad (24)$$

so that t_0 can be determined in terms of ε and τ_a . This has been done for various memory functions in [12].

The above argument is valid for zero temperature ($T = 0$) only. It may be generalised to non-zero temperatures by noting that fluctuations will initiate the transition even *before* $h_i \xi_i^\mu$ changes sign on I_μ . The reason is that for $T \neq 0$ ($\beta < \infty$) we have $\text{Prob}\{S_i(t + \Delta t) = \pm \xi_i^\mu\} = \frac{1}{2} [1 \pm \tanh[\beta h_i \xi_i^\mu]]$. Thus a substantial fraction of the spins is already flipped from ξ_i^μ to $\xi_i^{\mu+1} = -\xi_i^\mu$ and the transition starts, *once* $\beta h_i \xi_i^\mu$ decreases below a certain threshold B_0 on I_μ . Assuming, as before, that transition times are negligible and that $m_\nu(t) \sim 1$ while the system is in state ν , we conclude that, except for the difference in t_0 —which is to be determined—the right-hand side of (24) remains unaltered. The left-hand side of (24) must be replaced by the threshold $\beta^{-1} B_0$ so that we obtain

$$\beta^{-1} B_0 = 1 - \varepsilon + 2\varepsilon W_a(t_0, 2t_0). \quad (25)$$

An immediate consequence of (25) is that the *minimum* value of ε for which stable sequences occur has a β dependence of the form $\varepsilon_{\min}(\beta) = 1 - B_0/\beta$, which is not exact [15], but a reasonable approximation. Applying (25) to δ -function delay, we find that $t_0 = \tau_a$ for *all* $\varepsilon > \varepsilon_{\min}(\beta)$.

For exponential delay, (25) predicts t_0 to be a *continuous* function of ε and β , which diverges whenever ε approaches $\varepsilon_{\min}(\beta)$ from above. The value of B_0 is related

to a critical probability $\text{Prob}\{S_i(t + \Delta t) = -\xi_i^\mu\} = p^*$ for i in I_μ , which is the minimal fraction of spins that have to be flipped from ξ_i^μ to $\xi_i^{\mu+1} = -\xi_i^\mu$ on I_μ to induce the transition. Taking $p^* = 0.015$ or, equivalently $B_0 = 2.09$ reproduces the data rather well at low temperatures; see figure 3. Moreover, away from the critical boundary $\varepsilon_{\min}(\beta)$ the computed values of t_0 are rather insensitive to variations of the parameter p^* . This completes our discussion of the parameters relevant for the dynamics on the linear subparts of complex sequences.

The analysis of the dynamics at branch points relies on the same ideas. Returning to the song example of the previous sections, we compute the persistence time t^* in the state $(0, p_0)$. To do so, we assume that the system entered P_0 (the chorus) via P_γ , and that it has been in $(0, p_0)$ for a time δ , so that the local field at i is

$$h_i(t) = \xi_i^{0,p_0} + \varepsilon[\xi_i^{0,p_0} W_a(\delta, \delta + t_0) + \xi_i^{0,p_0-1} W_a(\delta + t_0, \delta + 2t_0) + \dots] + \tilde{\varepsilon} \xi_i^{\gamma+1,\gamma} W_a(0, \delta) W_b(\delta + (p_0 - 1)t_0, \delta + p_0 t_0). \tag{26}$$

Here we have used the setup of § 2 and assumed that τ_b is chosen so that $\bar{m}_{\gamma-1,p_{\gamma-1}}^b(t) \ll \bar{m}_{\gamma,p_\gamma}^b(t) = W_b = W_b(\delta + (p_0 - 1)t_0, \delta + p_0 t_0)$. Following the same line of reasoning as above, we conclude that the transition to $(\gamma + 1, 1)$ starts when $\beta h_i(t) \xi_i^{0,p_0}$ decreases below the threshold B_0 on I_{0,p_0} . Identifying the corresponding δ as t^* , we get

$$\beta^{-1} B_0 = 1 - \varepsilon + 2\varepsilon W_a(t^*, t^* + t_0) - W_a(0, t^*)[\tilde{\varepsilon} W_b - \varepsilon]. \tag{27}$$

To obtain (27), we have also assumed that $P_\gamma P_0$ lasted long enough to justify the replacement of $W_a(\delta + t_0, \delta + 2t_0) + W_a(\delta + 2t_0, \delta + 3t_0) + \dots$ by $1 - W_a(0, \delta + t_0)$ in (26). Once β , τ_a and ε , and thereby t_0 , are given, (27) relates t^* with τ_b and $\tilde{\varepsilon}$. If, in addition, we require $\delta = t^* = t_0$, we can use (25) and solve $\tilde{\varepsilon}$ in terms of ε and t_0 :

$$\tilde{\varepsilon} = \varepsilon / W_b = \varepsilon / W_b(p_0 t_0, (p_0 + 1)t_0). \tag{28}$$

With parameters as in figure 1, we compute $\tilde{\varepsilon}$ to be 16.59, whereas we had to take $\tilde{\varepsilon} = 17.75$. The discrepancy is mainly due to the short length of P_0 and to the differences between the measured and calculated values of t_0 (see figure 3).

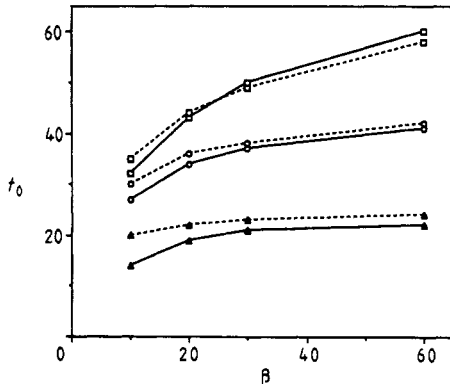


Figure 3. Persistence time t_0 for a standard sequence generator as a function of the inverse temperature β , for various values of the asymmetry parameter ε . The memory kernel w_a decays exponentially with $\tau_0 = 15$. Full lines represent theoretical values obtained from (23), broken lines 'measured' values obtained by solving (10) for a cycle of 9 patterns. Squares refer to $\varepsilon = 1$, circles to $\varepsilon = 1.1$ and triangles to $\varepsilon = 1.15$.

Similarly, with the two-neuron interactions of § 4 we find

$$\beta^{-1} B_0 = 1 - \varepsilon + 2\varepsilon W_a(t^*, t^* + t_0) + \varepsilon W_a(0, t^*) - \tilde{\varepsilon} W_b \quad (29)$$

and

$$\beta^{-1} B_0 = 1 - \varepsilon + 2\varepsilon W_a(t^*, t^* + t_0) + (\varepsilon - r\varepsilon_0) W_a(0, t^*) - \tilde{\varepsilon} W_b \quad (30)$$

for the versions without and with direct transition terms, respectively. Again, given β , τ_a and ε , the persistence time t^* may be computed as a function of $\tilde{\varepsilon}$ and τ_b (and, as the case may be, ε_0). Note that taking δ -function delay for w_b , with $\tau_b = (p_0 + 1)t_0$, bounds t^* in (29) and (30) by $t_0 \leq t^* < 2t_0$. Alternatively, one may prescribe t^* (within these bounds) and compute the appropriate value of $\tilde{\varepsilon}$ or of the allowed $(\tilde{\varepsilon}, \varepsilon_0)$ combinations. In all cases, we have found the predictions of (27)–(29) to agree well with our numerics.

The above discussion has been in terms of sequential dynamics. In the case of parallel dynamics, time is discrete, the dynamics is described by the iterative equation (10) instead of the differential equation (11), and the convolution integrals (7) must be replaced by infinite sums, with memory kernels adapted accordingly. In spite of all this, the final formulae for t_0 and t^* remain unaltered.

7. Complex sequences of correlated patterns

As it stands, the theory of complex association described above is applicable only to unbiased binary random patterns. An extension that works for *any* complex sequence of *linearly independent* patterns is, however, easily devised, using ideas of error-free recall of memories in networks with symmetric couplings [17, 23, 24]. To see this, consider a complex cycle of the form $P_1 P_0 P_2 P_0 \dots P_r P_0 \dots$ introduced in § 2. Let us denote by C the correlation matrix of the individual patterns of the complex cycle, with elements

$$C_{\rho, \mu; \sigma, \nu} = N^{-1} \sum_i \xi_i^{\rho, \mu} \xi_j^{\sigma, \nu} \quad (31)$$

For linearly independent patterns, its inverse C^{-1} exists, so that the following three types of synapse:

$$J_{ij}^{(1)} = N^{-1} \sum_{(\rho, \mu); (\sigma, \nu)} \xi_i^{\rho, \mu} (C^{-1})_{\rho, \mu; \sigma, \nu} \xi_j^{\sigma, \nu} \quad (32)$$

$$J_{ij}^{(2)} = \varepsilon N^{-1} \sum'_{(\rho, \mu)} \sum_{(\sigma, \nu)} \xi_i^{\rho, \mu} \otimes^1 (C^{-1})_{\rho, \mu; \sigma, \nu} \xi_j^{\sigma, \nu} \quad (33)$$

and in the three-neuron interaction approach of § 2:

$$J_{ijk}^{(3)} = \tilde{\varepsilon} N^{-2} \sum_{\gamma=1}^r \sum_{(\rho, \mu); (\sigma, \nu)} \xi_i^{\gamma+1, 1} (C^{-1})_{0, \rho_0; \rho, \mu} (C^{-1})_{\gamma, \rho; \sigma, \nu} \xi_j^{\rho, \mu} \xi_k^{\sigma, \nu} \quad (34)$$

of which the latter two are each endowed with the same dynamic response as above, will drive the system through the complex sequence, irrespective of the correlations among the patterns or their mean level of activity. To wit, expanding $S_i(t)$ in the manner of [24] according to

$$S_i(t) = \sum_{(\rho, \mu)} a_{\rho, \mu}(t) \xi_i^{\rho, \mu} + \delta S_i(t) \quad (35)$$

where $\{\delta S_i\}$ is orthogonal to all the patterns, and similarly for $\bar{S}_i^a(t)$ and $\bar{S}_i^b(t)$, we compute the local field to be

$$h_i(t) = \sum_{(\rho,\mu)} \xi_i^{\rho,\mu} a_{\rho,\mu}(t) + \varepsilon \sum_{(\rho,\mu)} \xi_i^{\rho,\mu \oplus 1} \bar{a}_{\rho,\mu}^a(t) + \tilde{\varepsilon} \sum_{\rho} \xi_i^{\rho+1,1} \bar{a}_{0,p_0}^a(t) \bar{a}_{\rho,p_\rho}^b(t). \quad (36)$$

To elucidate the formal equivalence with (9), we have again stayed with the essentially finite- q case. The above discussion of the dynamic features of the network may now be repeated word for word, with the $a_{\rho,\mu}(t)$ taking the role of the $m_{\rho,\mu}(t)$. The exact dynamic evolution equations are likewise easily rewritten, so as to take correlations into account.

8. Summary and discussion

To summarise, neural networks operating with multiple time delays were shown to be capable of complex forms of temporal association, including loop control. The underlying mechanisms of decision making on the basis of past experience of the network appear to be rather natural in view of omnipresent synaptic delays in neural networks [25]. Owing to the transparency of their implementation, these mechanisms easily allow for extensions which, if need be, should be capable of handling situations more complicated than those explicitly considered in the present paper; cf § 5. Furthermore, our theory also allows analytic treatment of complex sequences in neural networks with non-linear synapses [15]. Non-linear synapses, in particular clipped versions of the $J_{ij}^{(1)}$, $J_{ij}^{(2)}$ and $J_{ijk}^{(3)}$, are relevant to hardware realisations.

There is a potentially interesting application of the ideas put forward in the present paper, which we would like to mention. It combines methods for complex sequence generation with the capabilities of networks with non-linear synapses. Besides storing a certain set of patterns, non-linear synapses can also be used to store the results of *all* possible logical operations between pairs of these patterns as stable states of a neural net [22, 26]. These states can then serve as target or control states of transitions just as the states corresponding to the patterns themselves. They can, for instance, be connected to form a 'program' running on the net, which performs computations on patterns that go beyond simple pattern recognition, completion or association tasks.

A complementary approach to complex temporal association was recently proposed by Dehaene *et al* [10]. Their neural net differs from Hopfield-type networks in that its fundamental entities are *pattern-specific neural clusters*—resembling grandmother cells—which allow a continuum of activity levels but are otherwise of unspecified internal structure. The mechanism of temporal association—simple or complex—in their neural net is based on two- and three-cluster interactions (the synaptic triads [10]), with short-term synaptic plasticity during retrieval as an essential ingredient. Moreover, it appears that the sequences of Dehaene *et al* [10] are complex *only* at the level of the output clusters and that, taking internal clusters into account, the network as a whole does go through linear sequences.

In the present approach, short-term synaptic plasticity is dispensed with altogether. Instead, our fundamental concept is the control state, a special network state which initiates and controls not one but *several* transitions within the complex sequence, notably those where paths diverge. The corresponding transitions are encoded in synapses endowed with appropriate transmission delays. The introduction of multiple time delays is not *ad hoc*, as it may seem at first sight. On the contrary, a broad distribution of transmission delays is a natural aspect of brain morphology [27].

Moreover, it has been shown recently [20] that the coding of different transitions by synapses with different transmission delays in the manner envisaged in the present paper is a salient feature of Hebbian learning in formal neural networks operating with transmission delays.

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Note added in proof. After submission of this paper we became aware of recent work by Guyon *et al* [28] where related results, based on (non-local) exact storage prescriptions, were obtained for networks operating with parallel dynamics.

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